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On The Number of Representations of a Positive Integer by the Binary Quadratic Forms with Discriminants -128, -140

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Abstract
We shall obtain the exact formulas for the number of representations by primitive binury qvadratic forms with discriminants -128 and -140.
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I. Introduction

Let $f = f(x;y) = ax^2 + bxy + cy^2$ be a primitive integral positive-definite binary quadratic form. The positive integer *n* is said to be represented by the form *f* if there exists integers *x* and *y* such that $n = ax^2 + bxy + cy^2$.

The number of representations of *n* by the form *f* is denoted by $r(n; f)$. It is well known how to find the formulas for the number of representations of a positive integer by the positive-definite quadratic form which belong to one-class genera. Some papers are devoted to the case of multy-class genera. Using the simple theta functions Peterson [1] obtained formulas for $r(n; f)$ in the case of the binary forms with discriminant -44. These forms and some other ones were considered by P.Kaplan and k.S.Williams [2]. Their proof for odd number *n* based on Dirichlet theorem. In the same work in case of forms with discriminants equal to -80, -128 and -140 application of this theorem did not succeed and formulas only for even n have been received. In [3] we considered two binary forms $3x^2 + 2xy + 7y^2$ and $3x^2 - 2xy + 7y^2$ of discriminant -80 and two binary forms $3x^2 + 2xy + 11y^2$ and $3x^2 - 2xy + 11y^2$ of discriminant -128. Using Siegel's theorem [4] we obtained exact formulas for the number of representations by these forms. But in case of the other primitive forms with discriminants -128 and -140 we have to use the theory of modular forms. In this paper by means of the theory of modular forms the formulas for the number of representations of a positive integer by the forms $f_1 = x^2 + 32y^2$, $f_2 = 4x^2 + 4xy + 9y^2$, $f_3 = x^2 + 35y^2$, $f_4 = 4x^2 + 2xy + 9y^2$, $f_5 = 4x^2 - 2xy + 9y^2$, $f_6 = 5x^2 + 7y^2$, $f_7 = 3x^2 + 2xy + 12y^2$, $f_8 = 3x^2 - 2xy + 12y^2$ are obtained.

II. Basic results

In order to use the theory of modular forms in case of the binary forms $f_k (k = 1, 2, \dots 8)$) it is necessary to construct the cusp form $X(\tau)$ which is so-called remainder member. For this purpose we use the modular properties of the generalized theta – function defined in [5] as follows:

$$\vartheta_{gh}(\tau; p_\nu, f) = \sum_{x \equiv g(\mathrm{mod} N)} (-1)^{\frac{h'A(x-g)}{N^2}} p_\nu(x) e^{\frac{\pi i \tau x' A x}{N^2}}$$

Here *A* is an integral matrix of *f*, $x \in Z^S$, *g* and *h* are the special vectors with respect to the form *f*, $p_\nu(x)$ is a spherical function of the ν-th order corresponding to *f*; *N* is a step of the form *f*.

In particular, if f is a binary form, *g* and *h* are zero vectors and $p_0(x) = 1$, then

$\vartheta_{gh}(\tau; p_0, f) = \vartheta(\tau; f)$,

$r(n; f)$ is a Fourier coefficient of $\vartheta(\tau; f)$.

We assume, that

$\vartheta_{gh}(\tau; p_0, f) = \vartheta_{gh}(\tau; f)$, where $p_0 = 1$.

$E(\tau; f)$ is the Eisenstein series corresponding to *f* (see, e.g., [3]).

By means of the theory of modular forms we prove the following theorems.

Theorem 1.

Let $f_1 = x^2 + 32y^2$,

$f_2 = 4x^2 + 4xy + 9y^2$, $g = \begin{pmatrix} 16 \\ 0 \end{pmatrix}$,

$h=\begin{pmatrix}0\\2\end{pmatrix}$, $f=4x^2+8y^2$. Then we have

$$\vartheta(\tau;\ f_1)=\frac{1}{2}E(\tau;\ f_1)+\frac{1}{2}\vartheta_{gh}(\tau;\ f),$$

$$\vartheta(\tau;\ f_2)=\frac{1}{2}E(\tau;\ f_1)-\frac{1}{2}\vartheta_{gh}(\tau;\ f).$$

Theorem 2.

Let $f_3=x^2+35y^2$,

$f_4=4x^2+2xy+9y^2$

$f_5=4x^2-2xy+9y^2$,

$g=\begin{pmatrix}70\\0\end{pmatrix}$, $h=\begin{pmatrix}70\\0\end{pmatrix}$.

Then we have

$$\vartheta(\tau;\ f_3)=\frac{1}{2}E(\tau;\ f_3)+\frac{2}{3}\vartheta_{gh}(\tau;\ f_4)$$

$$\vartheta(\tau;\ f_4)=\vartheta(\tau;\ f_5)=\frac{1}{2}E(\tau;\ f_3)-\frac{1}{3}\vartheta_{gh}(\tau;\ f_4)$$

Theorem 3.

Let $f_6=5x^2+7y^2$, $f_7=3x^2+2xy+12y^2$,

$f_8=3y^2-2xy+12y^2$, $g=\begin{pmatrix}0\\70\end{pmatrix}$, $h=\begin{pmatrix}70\\0\end{pmatrix}$.

Then we have

$$\vartheta(\tau;\ f_6)=\frac{1}{2}E(\tau;\ f_6)-\frac{2}{3}\vartheta_{gh}(\tau;\ f_7),$$

$$\vartheta(\tau;\ f_7)=\vartheta(\tau;\ f_8)=\frac{1}{2}E(\tau;\ f_6)+\frac{1}{3}\vartheta_{gh}(\tau;\ f_7)$$

Equating the Fourier coefficients in both sides of the identities from theorems 1-3 we get the following theorems:

Theorem 4

Let $n=2^{\alpha}u$, $(u,\ 2)=1$, $f_1=x^2+32y^2$

$f_2=4x^2+4xy+9y^2$. Then

$$r(n;\ f_k)=\sum_{\nu|u}\left(\frac{-2}{\nu}\right)+\nu(n;\ f_k)$$ for $u\equiv 1(\mathrm{mod}\ 8)$,

$=2\sum_{\nu|u}\left(\frac{-2}{\nu}\right)$ for $\alpha=2$, $u\equiv 1(\mathrm{mod}\ 8)$ and for $\alpha>3$, $u\equiv 1,3(\mathrm{mod}\ 8)$,

$=0$ otherwise,

where $k=1,2$; $\left(\frac{-2}{\nu}\right)$ is Jakobi symbol and $\nu(n;\ f_k)=(-1)^{k-1}\frac{1}{2}\sum_{\substack{n=x^2+8y^2\\ 2\dagger x}}(-1)^y$

.

Theorem 5.

Let $n=2^{\alpha}5^{\beta}7^{\gamma}u$, $(u,10)=1$, $f_3=x^2+35y^2$,

$f_4=4x^2+2xy+9y^2$, $f_5=4x^2-2xy+9y^2$

Then

$$r(n;\ f_k)=\frac{1}{6}\left(1+(-1)^{\beta+\gamma}\left(\frac{u}{5}\right)\right)\Bigg(1+$$

$$+(-1)^{\beta+\gamma}\left(\frac{u}{7}\right)\Bigg)\sum_{\nu|u}\left(\frac{-35}{\nu}\right)+\nu(n;\ f_k)$$

For $\alpha=0$,

$$=\frac{1}{2}\left(1+(-1)^{\beta+\gamma}\left(\frac{u}{5}\right)\right)\Bigg(1+$$

$$+(-1)^{\beta+\gamma}\left(\frac{u}{7}\right)\Bigg)\sum_{\nu|u}\left(\frac{-35}{\nu}\right)$$ for $2|\alpha$, $\alpha>0$,

$=0$ for 2†α,

Where $k=3,4,5$; $\left(\frac{u}{5}\right)$, $\left(\frac{u}{7}\right)$, $\left(\frac{-35}{\nu}\right)$ are Jacobi symbols and

$$\nu(n;\ f_3)=\frac{2}{3}\sum_{\substack{n=x^2+xy+9y^2\\ 2\dagger x}}(-1)^y$$

$$\nu(n;\ f_4)=\nu(n;\ f_5)=-\frac{1}{3}\sum_{\substack{n=x^2+xy+9y^2\\ 2\dagger x}}(-1)^y.$$

Theorem 5.

Let $n=2^{\alpha}5^{\beta}7^{\gamma}u$, $(u,10)=1$, $f_6=5x^2+7y^2$, $f_7=3x^2+2xy+12y^2$,

$f_8=3x^2-2xy+12y^2$. Then

$$r(n;\ f_k)=\frac{1}{6}\left(1-(-1)^{\beta+\gamma}\left(\frac{u}{5}\right)\right)\Bigg(1-$$

$$-(-1)^{\beta+\gamma}\left(\frac{u}{7}\right)\Bigg)\sum_{\nu|u}\left(\frac{-35}{\nu}\right)+\nu(n;\ f_k)$$

for $\alpha=0$,

$$=\frac{1}{2}\left(1-(-1)^{\beta+\gamma}\left(\frac{u}{5}\right)\right)\Bigg(1-$$

$(-1)^{\beta+\gamma}\left(\frac{u}{7}\right)\Big)\sum_{v|u}\left(\frac{-35}{v}\right)$ for $2|\alpha$, $\alpha>0$

$=0$ for $2\dagger\alpha$,

where $k=6,7,8$; $\left(\frac{u}{5}\right)$, $\left(\frac{u}{7}\right)$, $\left(\frac{-35}{v}\right)$ are Jacobi symbols and

$$\nu(n;f_6)=-\frac{2}{3}\sum_{\substack{n=3x^2+xy+3y^2\\ 2\dagger y}}(-1)^x,$$

$$\nu(n;f_7)=\nu(n;f_8)=\frac{1}{3}\sum_{\substack{n=3x^2+xy+3y^2\\ 2\dagger y}}(-1)^x.$$

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